Reply to "Comment on 'Negative temperature of vortex motion' "

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We respond to the questions raised in the preceding Comment [K. O'Neil and L. J. Campbell, Phys. Rev. A **47**, 2966 (1993)]. We explain why our statement "vortex temperature is always positive" is equivalent to the "opposite" statement in the Comment: "vortex temperature is always negative," and why we viewed the negative temperature states as paradoxical. We discuss also the notion of temperature for the limit case of infinite number of vortices.

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Herein, we address the questions raised in the preceding Comment [1].

Sign of temperature. For reasons indicated in Refs. [2,3] we accept for a system of point vortices the Boltzmann temperature T based on the equipartition law for ergodic motions,

$$T=\left\langle x_1\frac{\partial H}{\partial x_1}\right\rangle=\cdots=\left\langle y_n\frac{\partial H}{\partial y_n}\right\rangle , \tag{1}$$

where (x_α,y_α) are coordinates of the αth vortex, H is the Hamiltonian, and $\langle\ \rangle$ means time average along a trajectory. It is clear that the sign of T depends on the sign of H. This ambiguity does not arise for "normal" systems [4] where H is identified with the sum of kinetic and potential energies, i.e., with the total energy. For such systems, the region $H\leq E$ in state space is generically homeomorphic to a ball with the canonical volume $\Gamma(E)$. With this assumption, the thermodynamic temperature $I/T=dS/dE$ is always positive if the entropy S is defined as $\ln\Gamma(E)$. This definition has two major advantages for systems with finitely many, even few, degrees of freedom: (a) S is adiabatically invariant and (b) thermodynamic and Boltzmann temperatures coincide.

The Hamiltonian system of point vortices is quite different from normal systems: H is not a sum of kinetic and potential energies. Furthermore, the choice of the sign for H depends on its interpretation as an interaction energy [5] or as a stream function [6–8]. Both Hamiltonians are dynamically equivalent and the boundary is attractive (repulsive) for the first (second) interpretation. At the same time, the stream function Hamiltonian only corresponds to a topological ball $H\leq E$ and leads to positive T, whereas the interaction energy H corresponds to a complement to the ball and leads to negative T as correctly indicated in the Comment. We conclude that for systems without preferable choice for the sign of H the statement that T does not change its sign is more meaningful than insisting that T is positive or negative.

Paradox of negative temperature. The very existence of negative temperature for some physical systems is a well understood phenomenon and is not now paradoxial. However, the change of sign of the Boltzmann temperature for vortex systems when H passes through a critical value $E_{\rm cr}$ seems really paradoxial. It follows from (1) that T is proportional to the averaged oriented area $\langle y_\alpha\dot{x}_\alpha\rangle$ bounded by the αth vortex trajectory. A change in the sign of T when H passes $E_{\rm cr}$ corresponds to the change of the average direction of rotation and such behavior of point vortices would be very paradoxial. We underline that this statement is related to the Boltzmann (and the corresponding thermodynamic) temperature, but not to the temperature adopted in [9–11].

Vortex clumping and ergodicity. We expressed a doubt [4] about clouds of positive and negative vortices being compatible with ergodicity of motion. The authors of the Comment argue that this is not the case. To be on more solid basis than using mainly intuitive arguments, we suggest that equipartition, which is a necessary condition for ergodicity, be checked for these motions. Such extensive numerical experiments are in progress now and hopefully will provide a definite answer to this question.

Limit of infinite number of vortices. This limit is very interesting and, perhaps, related to two-dimensional turbulence. The equation for average flow was obtained in Refs. [9–11], which received much attention especially in connection with the approach developed in Refs. [12–14]. Limit relations depend essentially on scaling [15,16]. It is natural to keep energy and total vorticity fixed while intensities k_α of individual vortices tend to zero as N^{-1}. This limit will be discussed in detail elsewhere; here we mention only a point relevant to the discussion here.

Limit relations can be studied using microcanonical and canonical distributions. For most systems, they are asymptotically equivalent. In particular, the canonical temperature $T_c=\beta^{-1}$ corresponding to the canonical distribution $f=Z^{-1}\exp(-\beta H)$ is equal to the Boltzmann temperature T_B. This is not the case for motion of vortices. Both T_B and T_c tend to zero as N^{-1} and should be normalized by the factor N. The limit values of normalized T_B and T_c coincide for $E > E_{cr}$ but are different (even have different signs) for $E < E_{cr}$, where E_{cr} is a critical value of H. The canonical temperature T_c is related to the usual entropy $S = \ln(d\Gamma/dE)$ while the Boltzmann temperature T_B is related to the adiabatic entropy $S = \ln\Gamma$. This explains why different expressions for entropy lead to conflicting results even in the case of $N \rightarrow \infty$. The system of point vortices is an interesting example for which (normalized) limit temperatures T_B and T_c do not coincide.

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